The Second Main Theorem on Complements Part 2

7/22/22

Strategy:

(BAB conjecture in dimension $\leq d$
"ε-lt log semi-Fanos (X,B) with X Fano type are bounded")

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(Canonical bundle formula & conjectures in dimension $\leq d$)

$\Longrightarrow$

> Bounded complements in dimension d

§1: Preliminaries

- (X,B) is <u>log semi-Fano</u> if

  $K_X + B$ is lc and $-(K_X+B)$ is nef $\overset{?}{\text{and}}$ big $\Big| \ \substack{[0,1] \\ \cup\!\mid}$

- Given a finite set of rational coefficients $\mathfrak{R}$,

  $$\Phi(\mathfrak{R}) := \left\{ 1 - \frac{r}{m} : m \in \mathbb{Z},\ m > 0,\ r \in \mathfrak{R} \right\} \cap [0,1]$$

  $$\overline{\mathfrak{R}} := \left\{ r_0 - m \sum_{i=1}^{s} (1 - r_i) : r_0, \ldots, r_s \in \mathfrak{R},\ m \in \mathbb{Z},\ m > 0 \right\} \cap \mathbb{R}_{\geq 0}$$

- $0 < \varepsilon' \leq \dfrac{1}{N_{d-1}(\overline{\mathfrak{R}}) + 2}$ "largest complement we need for dim. $d-1$"

<u>Reductions</u> (from last time):

$(X, B=\sum b_i B_i)$ klt log semi-Fano variety of dim. d,
X Fano type, $B \in \Phi(\mathcal{R})$

$$\Big\Downarrow \text{ log crepant } \mathbb{Q}\text{-factorial blowups}$$

X $\mathbb{Q}$-factorial, $\operatorname{discr}(X,B) > -1+\varepsilon'$

$B \in (\Phi(\mathcal{R}) \cup [1-\varepsilon', 1]) \cap \mathbb{Q}$

$D := \sum d_i B_i$, where $d_i = \begin{cases} 1, & b_i \geq 1-\varepsilon' \\ b_i, & \text{otherwise} \end{cases}$

Can prove: (X, D) is log canonical

$$\Big\Downarrow \text{ Run } -(K_X+D)\text{-MMP}$$

We obtain a $\mathbb{Q}$-factorial variety Y
with two boundaries: $B_Y = \sum b_i B_i$, $D_Y = \sum d_i B_i$
such that $\operatorname{discr}(Y, B_Y) > -1+\varepsilon'$, $B_Y \in (\Phi(\mathcal{R}) \cup [1-\varepsilon', 1]) \cap \mathbb{Q}$,
$D_Y \in \Phi(\mathcal{R})$, $D_Y \geq B_Y$, $d_i > b_i$ iff $d_i = 1$, $b_i \geq 1-\varepsilon'$.

Either

a) $\rho(Y) = 1$, $K_Y + D_Y$ ample, (Y, B_Y) klt log semi-Fano

b) (Y, D_Y) log semi-Fano, $\lfloor D_Y \rfloor \neq 0$

§2 The case $\rho = 1$

By way of contr., suppose $\exists$

$(X^{(m)}, B^{(m)})$ n_m-complemented

$n_m \longrightarrow +\infty$

$$(X^{(m)}, B^{(m)}) \xmapsto{\text{reductions}} (Y^{(m)}, B_Y^{(m)})$$

$\operatorname{discr}(Y^{(m)}) \geq \operatorname{discr}(Y^{(m)}, B^{(m)}) > -1 + \varepsilon'$

$\Rightarrow$ $Y^{(m)}$ bounded $\Rightarrow$ can assume $Y^{(m)} = Y$

By DCC on $\Phi(R)$, varieties

$B_i^{(m)}$ are bounded in degree w.r.t. some $Y \subseteq \mathbb{P}^N$

$\Rightarrow \quad \mathrm{Supp}\, B_y^{(m)} = \mathrm{Supp}\, B_y$

is constant (can assume this)

If mult's are $< 1-c$, done.

$\exists$ a seq. $B_y^{(m)} \longrightarrow B_y^{\infty}$, $\lfloor B_y^{\infty} \rfloor \neq 0$

$$B_y^{\infty} := \sum_i b_i^{\infty} B_i$$

<u>Claim</u>: $b_1^{\infty} = 1$ and all other $b_i^{\infty} \leq 1-\varepsilon'$.

<u>Pf</u>: $b_1^{\infty} = 1$, $\forall i$ $B_1 \cap B_i$ is codim. 2

& <u>non-empty</u> $\rho(Y) = 1$

$\Rightarrow \quad b_i^{\infty} \leq 1-\varepsilon'$ by the following lemma

<u>Lemma</u>: Let $(S \ni p, \Delta = \sum \lambda_i \Delta)$

be a log surf. germ. Assume

$\mathrm{discr}(S, \Delta) \geq -1+\varepsilon'$ at p.

Then $\sum \lambda_i \leq 2-\varepsilon'$.

<u>Pf</u>: choose $\pi : \underset{\substack{\uparrow \\ \text{smooth}}}{S'} \longrightarrow S$ étale away from p

$\Delta' = \pi^* \Delta$, $p' = \pi^{-1}(p)$. Then

$$\operatorname{discr}(S', \Delta') \geq \operatorname{discr}(S, \Delta) \geq -1+\varepsilon'$$

Blow up p': get E of discr.

$$1 - \sum \lambda_i \geq -1+\varepsilon'$$

Sketch of conclusion:

replace (Y, B_Y^∞) with $(\overset{\substack{\text{coeff. 1 in } B^\infty \\ \downarrow}}{B_1}, \operatorname{Diff}_{B_1}(B_Y^\infty - B_1))$

Use inductive hypothesis here,

get bounded comp. to $(B_1, \operatorname{Diff}_{B_1}(B_Y^\infty - B_1))$

$\underset{\text{extend}}{\Longrightarrow}$ comp. $K_Y + B^+$ of $K_Y + B_Y^\infty$

<u>Claim</u>: for $m \gg 0$, $D^{(m)} \leq B^+$

contradicting the fact that $\underbrace{K_Y + D_Y^{(m)}}$ ample

minus this is nef under hypothesis.

$\square$

§3 Canonical Bundle Formula

Case b) (Y, D_Y) log semi-Fano, $\lfloor D_Y \rfloor \neq 0$

Switch notation: $Y, B_Y, D_Y \rightsquigarrow X, B, D$

<u>Main idea:</u> $-(K_X + D)$ nef $\Rightarrow$

$-(K_X + D)$ semiample $\Rightarrow$ induces morphism

$f : X \to Z$

We'll look at the case $0 < \dim Z < \dim X$

X FT $\Rightarrow$ Z FT. We want to find

D_Z s.t.

$$K_X + D = f^*(K_Z + D_Z)$$

Then, the CBF & conj's show:

Coming up!

- coeff's of D_Z have uniform bound depending on $\dim X$, $\mathcal{R}$
- can find suitable D_Z so (Z, D_Z) is klt

$\Rightarrow$ by induction, (Z, D_Z) has bounded comp's

Suppose denom's of D_Z divide I, let $K_Z + D_Z^+$ be a n-comp. of $K_Z + D_Z$, $I \mid n$

$$H_Z := D_Z^+ - D_Z$$

Define: $$D^+ = D + f^* H_Z$$

Claim: $K_X + D^+$ is an n-comp. of $K_X + D$.

Pf: $$n(K_X + D^+) = (n/I)\, I\,(K_X + D) + (n/I)\, I\, f^* H_Z$$

$$\sim n/I\; f^* I(K_Z + D_Z) + n/I\; f^*(I H_Z)$$
$$= n/I\; f^*(I(K_Z + D_Z^+))$$

$$= f^*(n(K_Z + D_Z^+)) = f^*D \sim 0$$

Setup: Let $f: X \to Z$ be a contraction of normal var.'s, $K_X + D$ $\mathbb{Q}$-Cartier with the property $f^*L \sim_{\mathbb{Q}} K_X + D$ for some $\mathbb{Q}$-Cartier L on Z.

Suppose (X, D) lc near generic fiber of f.

Structure of CBF:

$$K_X + D \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z)$$

discriminant of f

moduli part of f

Given a divisor $W \subseteq Z$, the divisorial pullback $f^\bullet W$ is the closure of $f^* W$ over $Z \setminus V$ where $\operatorname{codim}(V) \geq 2$, $V \supset \operatorname{Sing} Z$, $f: X \to Z$

is equidim over $Z \setminus V$

<u>Def</u>: For a prime divisor $W \subseteq Z$, let

$$c_W := \sup\{c : (X, D + c f^* W) \text{ is lc over the generic point of } W\}$$

$$B_Z := \sum_W (1 - c_W) W.$$

Choose minimal $r \in \mathbb{Z}_+$ and rat'l fn $\varphi \in k(X)$ s.t. $K_X + D + \frac{1}{r} = f^* L$

The <u>moduli part</u> of f is

$$M_Z : L - K_Z - B_Z$$

<u>Ex</u>:

1) $f : X \to Z$ is birational

$D = \sum d_i D_i$. Then: $B_Z = f_* D$

$M_Z = 0$.

$\Rightarrow \quad K_X + D = f^*(K_Z + f_* D)$

Reason: given $W \subseteq Z$,

f is an iso. over gen. point of W, so $(X, D + c f^{\bullet} W)$

lc over the gen. point of W

means: $c \leq 1$ if $f^{\bullet} W$ is not a comp. of D

$c \leq 1 - d_i$ if $f^{\bullet} W$ is a comp. of D

$$\Rightarrow \quad B_Z = f_* D$$

$$M_Z = 0$$

2) Elliptic fibrations over curves

a) Hyperelliptic surfaces

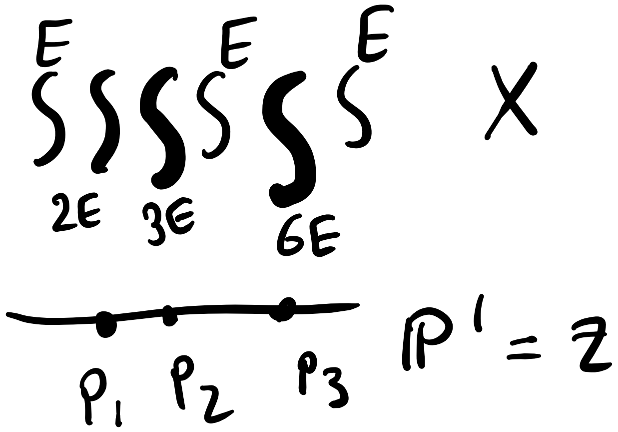
$$X = (E \times C)/G \leftarrow \text{finite gp.}$$

$\uparrow \nearrow$ sm. elliptic curves

$G \curvearrowright E$ translation

$G \curvearrowright C$ by elliptic curve auto's

$f : X \to Z = C/G = \mathbb{P}^1$



Claim: B_Z

$= \frac{1}{2}p_1 + \frac{2}{3}p_2 + \frac{5}{6}p_3$

$f^* p_1 = 2E$

$M_Z \sim_{\mathbb{Q}} 0$ here too: $K_Z + B_Z \sim_{\mathbb{Q}} 0$

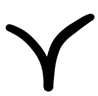
$K_X \sim_{\mathbb{Q}} 0$

b) General elliptic fibrations

$f: X \to Z$ elliptic fibr

X: rel. min. $/Z$; Z: sm. curve

B_Z can be read off from types of sing. fibers

 $\rightsquigarrow \frac{1}{3}$

$12 M_Z = j^* \mathcal{O}_{\mathbb{P}^1}(1)$, where $j: Z \to \mathbb{P}^1$ is the j-invariant

CBF conjectures

$$K_X + D = f^*(K_Z + B_Z + M_Z)$$

1) M_Z is semiample (after modification of Z)

2) Let X_η be the generic fiber of f.

Then $I_0(K_{X_\eta} + D_\eta) \sim 0$

where I_0 depends only on $\dim X_\eta$ mult's of (horizontal part) of D

follows from inductive hypothesis in our setup

3) M_Z is effectively semiample: there is an I depending only on $\dim X$, D, so that $I M_Z$ is basepoint free.